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DISCRIMINATION, INCOME AND WEALTH DISTRIBUTION, AND BUSINESS CYCLES

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Abstract:
This study generalizes the heterogeneous-household growth model with dynamic interdependence between economic growth, inequality in wealth and income, and discrimination proposed by Zhang (2017). The original model is in an integration of Walrasian-general-equilibrium theory and neoclassical-growth theory under influences of literature of economic discrimination. The economy is composed of one capital good sector, one consumer good sector, and multiple groups of households. It describes a dynamic interdependence between wealth accumulation, income and wealth distribution, time distribution and division of labor under discrimination. The model is generalized so that all constant parameters in the original model are time-dependent. The generalization makes the original model more robust as the current model allows us to study effects of exogenous time-dependent perturbations on the movement of the system. We provide a few examples of business cycles due to periodic exogenous shocks.

Key words: Business cycle; Discrimination; Neoclassical growth theory; Walrasian general equilibrium theory; Income transfer; inequality.

JEL classification: O12, E13, J71

1. INTRODUCTION

Discrimination is a well-observed phenomenon in different societies in different aspects. The importance of study on discrimination, economic mechanisms and consequences has been emphasized is recognized by some economists (e.g., Becker, 1957; Welch, 1967, 1975; Phelps, 1972, Borjas, 1992; Whatley and Wright, 1994; Carneiro, *et al*. 2005; Shi, 2006; Gabriel and Schmitz, 2014). Arrow (1998) describes the situation, which is still valid today as follows: "Racial discrimination pervades every aspect of a society in which it is found. It is found above all in attitudes of both groups, but also in social relations, in intermarriage, in residential location, and, frequently, in legal barriers. It is also found in levels of economic accomplishment; that is, income, wages, prices paid, and credit extended. This economic dimension hardly appears in general treatments of economics, outside of the specialized literature devoted to it." In particular, there are few theoretical models on interdependence between discrimination, income and wealth discrimination, and economic growth. Recently Zhang (2017) proposed a heterogeneous-household growth model with discrimination. This study is to further develop Zhang's model by allowing constant parameters to be time-dependent. This makes the original model more robust. Zhang's model is built by integrating the Walrasian general equilibrium theory (e.g., Walras, 1874; Arrow and Hahn, 1971; and Mas-Colell *et al.*, 1995) and the neoclassical growth theory (Solow, 1956; Takayama, 1997). It examines interdependence between discrimination, economic growth and inequality in income and wealth. The paper is organized as follows. Section 2 makes Zhang's model more general and robust by making all constant parameters to be time-dependent. Section 3 examines dynamic properties of the system and simulates the model for the special case given in Zhang's model. Section 4 shows existence of business cycles with different periodic exogenous shocks. Section 5 concludes the study.

2. THE BASIC MODEL

This section makes all the constant parameters Zhang (2017) to be time-dependent. Further explanation about the model refers to the original model. The model of production side follows the two-sector Uzawa growth model (Uzawa, 1961; and Takayama, 1997). Capital good and consumer good are supplied by capital good sector and consumer good sector, respectively. Capital is depreciated at time-dependent exponential rate $\delta_k(t)$. The assets of the economy is held by household who distribute their incomes to consume and save. Households undertake saving. Factors are fully utilized at every moment. There are J types of household. Households are homogenous within each group and vary between groups in regards to human capital, preference and social power. A group may be discriminated by, discriminate, or have "neutral" relations with other groups. A group's population is represented by $\bar{N}_j(t)$, ($j = 1, ..., J$). The price of the commodity is chosen unity. The other prices are measured in terms of capital good. The wage rate of group j and rate of interest are denoted by $w_j(t)$ and $r(t)$, respectively. We have $K(t)$ for total capital stock. Let index i and s represent for capital good and consumer good sector. We use $N_j(t)$ and $K_j(t)$ to stand for labor force and capital stocks employed by sector j. We apply $\bar{T}_j(t)$ and $T_j(t)$ to stand for leisure time and work time for the household of group j. The total labor supply $N(t)$ of the economy consists of the labor inputs of all the groups as follows:

$$N(t) = \sum_{j=1}^{J} h_j(t) T_j(t) \bar{N}_j(t). \tag{1}$$

We introduce

$$k_q(t) \equiv \frac{K_q(t)}{N_q(t)}, \;\; n_q(t) \equiv \frac{N_q(t)}{N(t)}, \;\; k(t) \equiv \frac{K(t)}{N(t)}, \;\; q = i, s.$$

We express the assumption of full employment of labor as follows:

$$N_i(t) + N_s(t) = N(t). \tag{2}$$

Capital good sector

We express the production function $F_i(t)$ as follows:

$$F_i(t) = A_i(t) K_i^{\alpha_i(t)}(t) N_i^{\beta_i(t)}(t), \;\; A_i(t), \alpha_i(t), \beta_i(t) > 0, \;\; \alpha_i(t) + \beta_i(t) = 1, \tag{3}$$

in which $A_i(t)$, $\alpha_i(t)$ and $\beta_i(t)$ are time-dependent parameters. The marginal conditions for optimizing profits mean

$$r(t) + \delta_k(t) = \alpha_i(t) A_i(t) K_i^{-\beta_i(t)}(t) N_i^{\beta_i(t)}(t), \;\; w_j(t) = h_j(t) w(t), \tag{4}$$

in which

$$w(t) \equiv \beta_i(t) A_i(t) K_i^{\alpha_i}(t) N_i^{-\alpha_i(t)}(t).$$

Consumer good sector

The production function is

$$F_s(t) = A_s(t) K_s^{\alpha_s(t)}(t) N_s^{\beta_s(t)}(t), \;\; A_s(t), \alpha_s(t), \beta_s(t) > 0, \;\; \alpha_s(t) + \beta_s(t) = 1. \tag{5}$$

We have the following marginal conditions:

$$r(t) + \delta_k(t) = \alpha_s(t)p(t)A_s(t)K_s^{-\beta_s(t)}(t)N_s^{\beta_s(t)}(t),$$
$$w_j(t) = \beta_s(t)h_j(t)p(t)A_s(t)K_s^{\alpha_s(t)}(t)N_s^{-\alpha_s(t)}(t), \quad (6)$$

in which $p(t)$ is the price of consumer good.

Consumer behaviors

This study applies Zhang's utility function and concept of disposable income to model consumer behavior. We use $\bar{k}_j(t)$ to present the household's wealth. This implies $\bar{k}_j(t) = \bar{K}_j(t)/\bar{N}_j(t)$, where $\bar{K}_j(t)$ is the total wealth owned by group j. We use $\varphi_j(t)$ to represent the lump sum transfer that group $j's$ household gets from conducting discrimination. If the group is discriminated, the following holds: $\varphi_j(t) \le 0$. If the group discriminates, the following holds: $\varphi_j(t) > 0$. The household's current income from the interest payment and the wage payment is given as follows:

$$y_j(t) = r(t)\bar{k}_j(t) + (1 - \phi_{wj}(t))T_j(t)w_j(t) + \phi_j(t),$$

where $\phi_{wj}(t)$ is the discrimination rate against group j in wage income. If there is no discrimination, then $\phi_{wj}(t) = 0$; otherwise $\phi_{wj}(t) > 0$. We use $\phi_{kj}(t)$ to stand for the discrimination rate on wealth against group j. The household disposable income composes the current disposable income and the value of net wealth

$$\hat{y}_j(t) = y_j(t) + (1 - \phi_{kj}(t))\bar{k}_j(t). \quad (7)$$

The disposable income is used up for saving and consumption.

Household j distribute the disposable income budget between savings $s_j(t)$ and consumption of consumer good $c_j(t)$. The discrimination rate on group $j's$ consumption is symbolized by $\phi_{cj}(t)$. We have the following the budget constraint:

$$(1 + \phi_{cj}(t))p(t)c_j(t) + s_j(t) = \hat{y}_j(t). \quad (8)$$

Let leisure time be represented by $\bar{T}_j(t)$ and (fixed) available time for work and leisure by T_0. Each one is faced with the following time constraint:

$$T_j(t) + \bar{T}_j(t) = T_0. \quad (9)$$

Insert (9) in (8)

$$(1 - \phi_{wj}(t))w_j(t)\bar{T}_j(t) + (1 + \phi_{cj}(t))p(t)c_j(t) + s_j(t) = \bar{y}_j(t), \quad (10)$$

in which

$$\bar{y}_j(t) \equiv (r(t) + 1 - \phi_{kj}(t))\bar{k}_j(t) + (1 - \phi_{wj}(t))T_0\, w_j(t) + \phi_j(t).$$

Utility functions are specified as

$$U_j(t) = \bar{T}_j^{\sigma_{0j}(t)}(t) c_j^{\xi_{0j}(t)}(t) s_j^{\lambda_{0j}(t)}(t), \quad \sigma_{0j}(t),\ \xi_{0j}(t),\ \lambda_{0j}(t) > 0,$$

in which $\sigma_{0j}(t)$ stands for the propensity to use leisure time, $\xi_{0j}(t)$ the propensity to consume consumption goods, and $\lambda_{0j}(t)$ the propensity to own wealth. We maximize the utility subject to (10) to get the following first-order conditions:

$$w_j(t)\bar{T}_j(t) = \sigma_j(t)\bar{y}_j(t), \quad p(t)c_j(t) = \xi_j(t)\bar{y}_j(t), \quad s_j(t) = \lambda_j(t)\bar{y}_j(t), \tag{11}$$

where

$$\sigma_j(t) \equiv \frac{\rho_j(t)\sigma_{0j}(t)}{1-\phi_{wj}(t)}, \quad \xi_j(t) \equiv \frac{\rho_j(t)\xi_{0j}(t)}{1+\phi_{cj}(t)}, \quad \lambda_j(t) \equiv \rho_j(t)\lambda_{0j}(t),$$
$$\rho_j(t) \equiv \frac{1}{\sigma_{0j}(t) + \xi_{0j}(t) + \lambda_{0j}(t)}.$$

The change in wealth is saving minus dissaving

$$\dot{\bar{k}}_j(t) = s_j(t) - \bar{k}_j(t) = \lambda_j(t)\bar{y}_j(t) - \bar{k}_j(t). \tag{12}$$

Balance of demand and supply
The equilibrium condition in consumer good markets is

$$\sum_{j=1}^{J} c_j(t)\bar{N}_j(t) = F_s(t). \tag{13}$$

Output of capital good sector equals depreciation of capital stock and net saving

$$S(t) - K(t) + \delta_k(t)K(t) = F_i(t), \tag{14}$$

in which

$$S(t) \equiv \sum_{j=1}^{J} s_j(t)\bar{N}_j(t), \quad K(t) = \sum_{j=1}^{J} \bar{k}_j(t)\bar{N}_j(t).$$

Full employment of capital implies

$$K_i(t) + K_s(t) = K(t). \tag{15}$$

Transfers between groups due to discrimination
We first define the total income from discrimination. In our approach, it equals the total income $\Gamma_w(t)$ from discrimination in wage incomes, the total income $\Gamma_c(t)$ from discrimination in consumer good market, and the total income $\Gamma_k(t)$ from discrimination in wealth. We have $\Gamma_w(t)$, $\Gamma_c(t)$ and $\Gamma_k(t)$, respectively, as follows:

$$\Gamma_w(t) = \sum_{j=1}^{J} \phi_{wj}(t) T_j(t) w_j(t) \bar{N}_j(t), \ \Gamma_c(t) = \sum_{j=1}^{J} \phi_{cj} \, p(t) c_j(t) \bar{N}_j(t), \ \Gamma_k(t) = \sum_{j=1}^{J} \phi_{kj}(t) \bar{k}_j(t) \bar{N}_j .$$

The discriminating groups share the total income

$$\Gamma_w(t) + \Gamma_c(t) + \Gamma_k(t) = \sum_{j=1}^{J} \phi_j(t) \bar{N}_j(t).$$

Per household lump sum transfers $\phi_j(t)$ are assumed equal among the discriminating groups. We use variable $\delta_j(t) = 1$ if j is a discriminating group, and $\delta_j = 0$ otherwise. We have

$$\phi_j(t) = \delta_j(t) \phi(t).$$

From the definition, we have

$$\phi(t) = \frac{\Gamma_w(t) + \Gamma_c(t) + \Gamma_k(t)}{N_0(t)}, \tag{16}$$

where $N_0(t) \equiv \sum_0^1 \delta_j(t) \bar{N}_j(t)$.

We competed the model.

3. THE DYNAMICS AND ITS PROPERTIES

As the dimension of dynamics of the economy equals the number of different groups, the dynamic analysis of nonlinear economy is generally complicated. Nevertheless, we can simulate movement of the system with computer. Before we show how to compute dynamics of the system, we introduce

$$z(t) \equiv \frac{r(t) + \delta_k(t)}{w(t)}, \ \{\bar{k}_j(t)\} \equiv (\bar{k}_2(t), \cdots, \bar{k}_J(t)).$$

Lemma

The motion of the economic system is given by J differential equations with $z(t)$, $\{\bar{k}_j(t)\}$, and t as variables

$$\dot{z}(t) = \Lambda_1(z(t), \{\bar{k}_j(t)\}, t),$$
$$\dot{\bar{k}}_j(t) = \Lambda_j(z(t), \{\bar{k}_j(t)\}, t), \quad j = 2, ..., J, \tag{17}$$

where $\Lambda_j(t)$ are dependent on $z(t)$, $\{\bar{k}_j(t)\}$, and t as shown in the Appendix. We have the other variables as functions of $z(t)$, $\{\bar{k}_j(t)\}$, and t determined by the following procedure: $\bar{k}_1(t)$ by (A21) → $r(t)$ and $w_j(t)$ with (A3) → $\phi(t)$ from (A20) → $\bar{y}_j(t)$ from (A4) → $N(t)$ from (A13) → $K_i(t)$ and $K_s(t)$ with (A15) → $N_i(t)$ and $N_s(t)$ with (A1) → $F_i(t)$ from (3) → $F_s(t)$ from (5) → $p(t)$ from (A8) → $\bar{T}(t)_j$, $c_j(t)$, and $s_j(t)$ with (11) → $T_j(t) = T_0(t) - \bar{T}_j(t)$ → $K(t) = K_i(t) + K_s(t)$.

This lemma provides a computational procedure for computing movement of the economic system. In order to show effects of exogenous perturbations, we first simulate the model with constant parameters. The rest of this section summarizes the results in Zhang (2017). The parameter values are as follows:

$$A_i = 1.3, \ A_s = 1, \ \alpha_i = 0.29, \ \alpha_s = 0.32, \ T_0 = 1, \ \delta_k = 0.05, \ \delta_1 = 1, \ \delta_2 = \delta_3 = 0,$$
$$\phi_{q1} = \phi_{q2} = 0, \ \phi_{q3} = 0.05, \ q = w, c, k.$$
$$\begin{pmatrix} N_1 \\ N_2 \\ N_3 \end{pmatrix} = \begin{pmatrix} 50 \\ 300 \\ 200 \end{pmatrix}, \ \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 0.6 \end{pmatrix}, \ \begin{pmatrix} \xi_{10} \\ \xi_{20} \\ \xi_{30} \end{pmatrix} = \begin{pmatrix} 0.12 \\ 0.16 \\ 0.18 \end{pmatrix}, \ \begin{pmatrix} \lambda_{10} \\ \lambda_{20} \\ \lambda_{30} \end{pmatrix} = \begin{pmatrix} 0.78 \\ 0.75 \\ 0.7 \end{pmatrix}, \ \begin{pmatrix} \sigma_{10} \\ \sigma_{20} \\ \sigma_{30} \end{pmatrix} = \begin{pmatrix} 0.25 \\ 0.18 \\ 0.15 \end{pmatrix}. \quad (18)$$

The initial conditions are

$$z(0) = 0.048, \ k_{26}(0) = 6, \ \bar{k}_3(0) = 1.5.$$

The movement of the economic system is given in Figure 1.

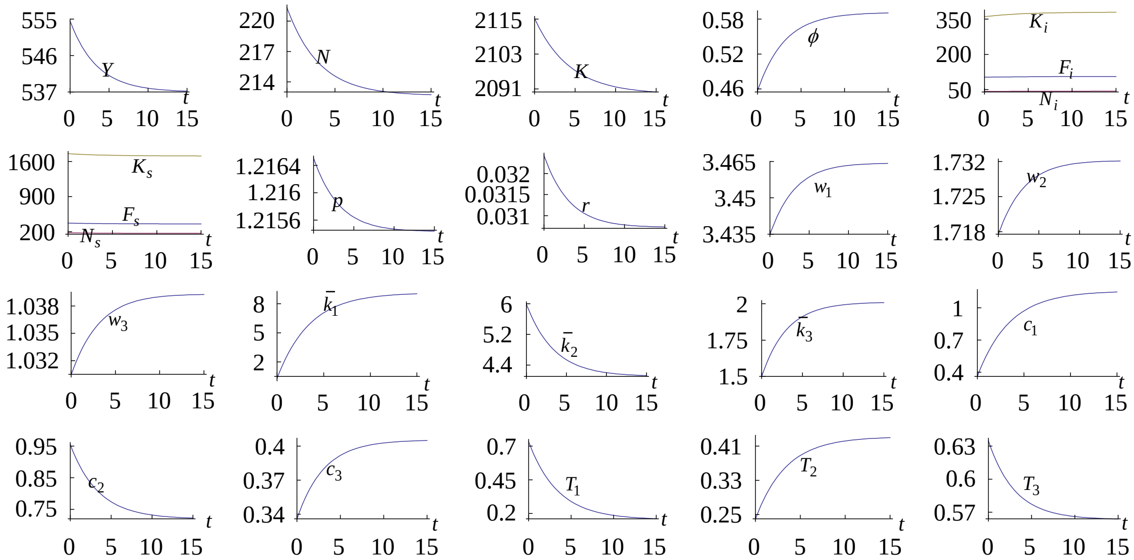

Figure 1. The Motion of the Economic System

We list the equilibrium values as follows:

$$Y = 536.97, \ \phi = 0.59, \ F_i = 105.77, \ F_s = 354.77, \ w_1 = 3.46, \ w_2 = 1.73, \ w_3 = 1.04,$$
$$r = 0.031, \ p = 1.22, \ N = 212.64, \ N_i = 43.36, \ N_s = 169.28, \ K = 2089.17, \ K_i = 380,$$
$$K_s = 1709.2, \ \bar{k}_1 = 9.14, \ \bar{k}_2 = 4.1, \ \bar{k}_3 = 2.01, \ T_1 = 0.154, \ T_2 = 0.432, \ T_3 = 0.563,$$
$$c_1 = 1.16, \ c_2 = 0.72, \ c_3 = 0.41. \quad (21)$$

The three eigenvalues are:

$$\{-0.34, \ -0.30, \ -0.23\}.$$

The system has a unique equilibrium point. It has three real negative eigenvalues. The stability guarantees the validity of the following comparative dynamic analysis.

4. COMPARATIVE DYNAMIC ANALYSIS

The previous section showed the movement of the three-group economy with fixed discrimination rates. The special case was provided by Zhang (2017). This section shows existence of business cycles due to different exogenous periodic shocks. We use $\bar{\Delta} x_j(t)$ to stand for the change rate of the variable, $x_j(t)$, in percentage due to changes in parameter value.

4.1. THE DISCRIMINATION RATE ON WAGE INCOME PERIODICALLY OSCILLATES

We now deal with the case that the discrimination rate on wage income oscillates in the following way:

$$\phi_{w3}(t) = 0.05 + 0.05\sin(t).$$

Figure 2 plots the simulation result. All variables oscillate periodically due to the shocks. The per household income from discrimination oscillates greatly.

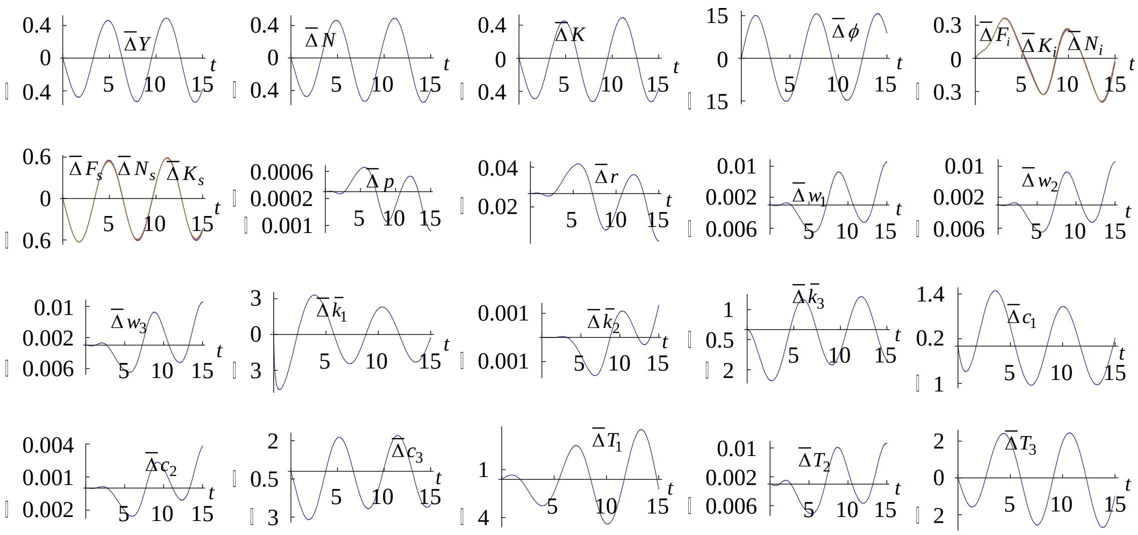

Figure 2. The Discrimination Rate on Wage Income Periodically Oscillates

4.2. THE DISCRIMINATION RATE ON WEALTH CHANGES PERIODICALLY

We study the case that the discriminator changes periodically the discrimination rate on wealth as follows:

$$\phi_{k3}(t) = 0.05 + 0.05\sin(t).$$

The simulation result is given Figure 3.

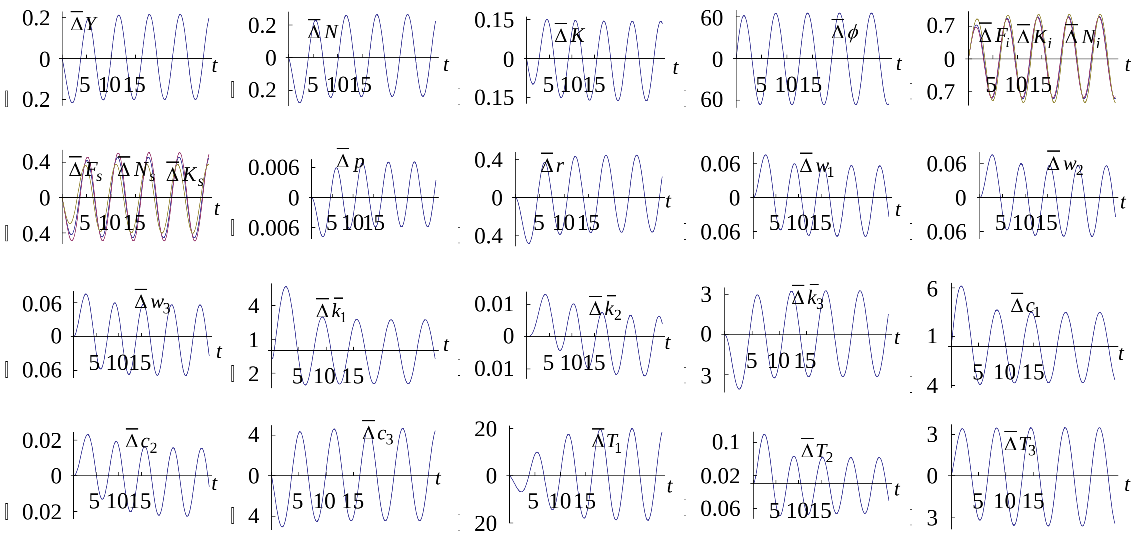

Figure 3. The Discrimination Rate on Wealth Changes Periodically

4.3. THE DISCRIMINATION RATE ON CONSUMPTION CHANGES PERIODICALLY

We now confirm existence of business cycles when the discrimination rate on consumption changes periodically as follows:

$$\phi_{c3} = 0.05 + 0.05\sin(t).$$

The simulation result is given in Figure 4.

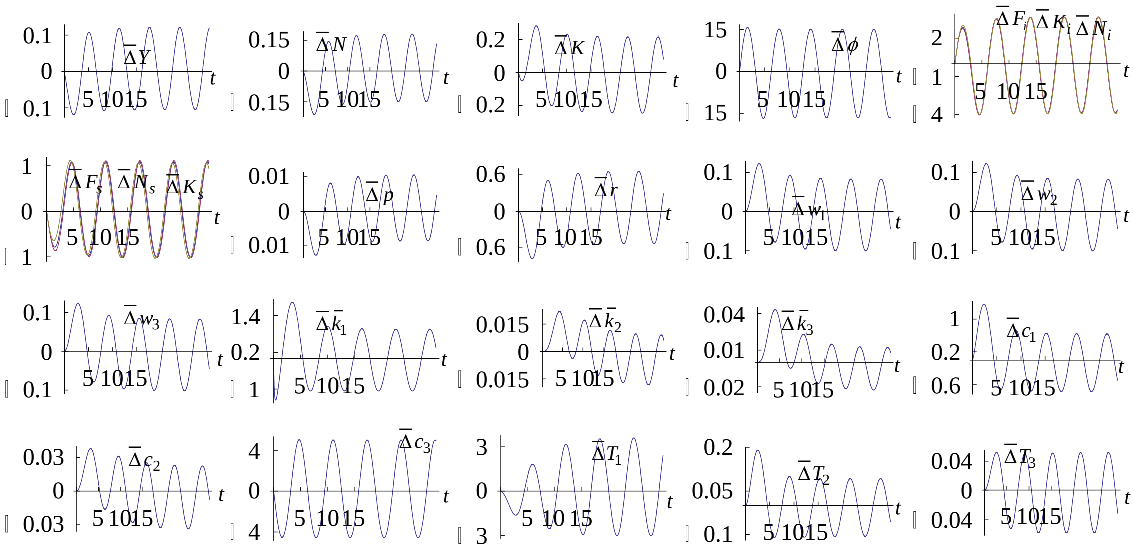

Figure 4. The Discrimination Rate on Consumption Changes Periodically

4.4. THE DISCRIMINATED PEOPLE'S HUMAN CAPITAL OSCILLATES

We show what happen to the economic system if the discriminated group's human capital oscillates as follows:

$$h_3(t) = 0.6 + 0.1\sin(t).$$

The simulation result is given in Figure 5.

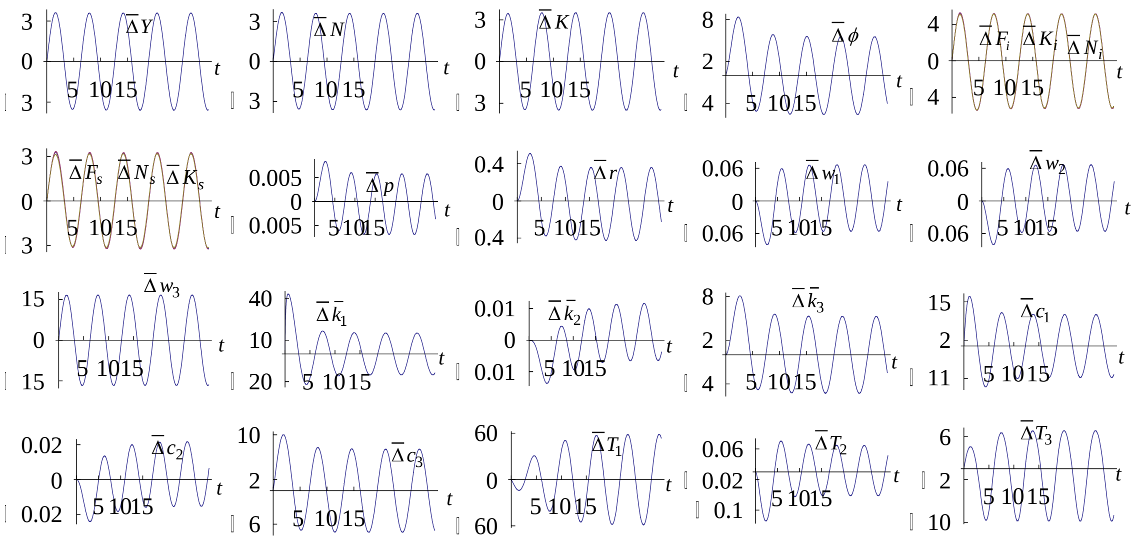

Figure 5. The Discriminated People's Human Capital Oscillates

5. CONCLUDING REMARKS

This study generalized the heterogeneous-household growth model with dynamic interactions between economic growth, inequality in wealth and income, and discrimination proposed by Zhang (2017). The original model is an integration of neoclassical-growth theory and Walrasian-general-equilibrium theory under influences of literature of economic discrimination. The economy is composed of one capital good sector, one consumer good sector, and multiple groups of households. The model describes a dynamic interdependence between endogenous accumulation change, economic structural change, income and wealth distribution, and division of labor under discrimination. Our generalization is to allow all constant parameters to be time-dependent. The generalization makes the original model more robust as the current model allows us to study effects of almost any types of exogenous time-dependent perturbations on movement of the system. We provided a few examples of business cycles due to periodic exogenous shocks. It is straightforward to simulate almost any type of time-dependent exogenous shocks. We may also extend the model in different ways. For instance, discrimination rate is reasonably an endogenous variable. Further studies on endogenous discrimination might deepen our understanding of dynamics of interdependence of different ethnic groups over time and space.

Appendix: Proving the Lemma

Equations (4) and (6) imply

$$z \equiv \frac{r + \delta_k}{w_j / h_j} = \frac{N_i}{\bar{\beta}_i K_i} = \frac{N_s}{\bar{\beta}_s K_s}, \tag{A1}$$

where $\bar{\beta}_j \equiv \beta_j / \alpha_j$. Substitute (A1) into (2)

$$\bar{\beta}_i K_i + \bar{\beta}_s K_s = \frac{N}{z}. \tag{A2}$$

Substitute (A1) into (4)

$$r = \alpha_r z^{\beta_i} - \delta_k, \quad w_j = \alpha_j z^{-\alpha_i}, \tag{A3}$$

in which

$$\alpha_r = \alpha_i A_i \bar{\beta}_i^{\beta_i}, \quad \alpha_j = h_j \beta_i A_i \bar{\beta}_i^{-\alpha_i}.$$

From the definitions of $\bar{y}_j$ and (A3), we obtain

$$\bar{y}_j = g_j \bar{k}_j + \bar{g}_j + \delta_j \phi, \tag{A4}$$

where

$$g_j(z) \equiv \alpha_r z^{\beta_i} - \delta_k + 1 - \phi_{kj}, \quad \bar{g}_j(z) \equiv (1 - \phi_{wj}) T_0 \alpha_j z^{-\alpha_i}.$$

Substitute $p c_j = \xi_j \bar{y}_j$ into (13)

$$\sum_{j=1}^{J} \xi_j \bar{N}_j \bar{y}_j = p F_s. \tag{A5}$$

Insert (A4) in (A5)

$$\sum_{j=1}^{J} \tilde{g}_j \bar{k}_j = p F_s - g - \xi_0 \phi, \tag{A6}$$

where

$$\tilde{g}_j(z) \equiv \xi_j \bar{N}_j g_j, \quad g(z) \equiv \sum_{j=1}^{J} \xi_j \bar{N}_j \bar{g}_j, \quad \xi_0 \equiv \sum_{j=1}^{J} \delta_j \xi_j \bar{N}_j.$$

Equations (4) and (6) imply

$$r + \delta_k = \alpha_i A_i K_i^{-\beta_i} N_i^{\beta_i} = \alpha_s p A_s K_s^{-\beta_s} N_s^{\beta_s}. \tag{A7}$$

Substitute (A1) into (A7)

$$p = \frac{\alpha_i A_i \bar{\beta}_i^{\beta_i} z^{\beta_i - \beta_s}}{\alpha_s A_s \bar{\beta}_s^{\beta_s}}. \tag{A8}$$

From (6), we get

$$p F_s = \frac{w_1 N_s}{h_1 \beta_s}. \tag{A9}$$

Equations (A9) and (A1) imply

$$p\,F_s = \frac{\bar{\beta}_s\, w_1\, z\, K_s}{h_1\, \beta_s}. \tag{A10}$$

Substitute (A10) into (A6)

$$\sum_{j=1}^{J} \tilde{g}_j\, \bar{k}_j = g_0\, K_s - g - \xi_0\, \phi, \tag{A11}$$

where

$$g_0(z) \equiv \frac{\bar{\beta}_s\, w_1\, z}{h_1\, \beta_s}.$$

Applying (1) and (9), we obtain

$$N = T_0 \sum_{j=1}^{J} h_j\, \bar{N}_j - \sum_{j=1}^{J} \frac{h_j\, \sigma_j\, \bar{y}_j\, \bar{N}_j}{w_j}, \tag{A12}$$

in which we also use $w_j\, \bar{T}_j = \sigma_j\, \bar{y}_j$. Substitute (A4) into (A12)

$$N = \tilde{\varphi}_0 - \sum_{j=1}^{J} \tilde{\varphi}_j\, \bar{k}_j - \hat{\varphi}\, \phi, \tag{A13}$$

where

$$\tilde{\varphi}_0(z) \equiv \sum_{j=1}^{J} \left(T_0 - \frac{\sigma_j\, \bar{g}_j}{w_j} \right) h_j\, \bar{N}_j, \;\; \tilde{\varphi}_j(z) \equiv \frac{h_j\, \sigma_j\, \bar{N}_j\, g_j}{w_j}, \;\; \hat{\varphi} \equiv \sum_{j=1}^{J} \frac{\delta_j\, h_j\, \sigma_j\, \bar{N}_j}{w_j}.$$

From (15), we get

$$K_i + K_s = K = \sum_{j=1}^{J} \bar{k}_j\, \bar{N}_j. \tag{A14}$$

With (A2) and (A14), we get

$$K_i = \beta\, \bar{\beta}_s \sum_{j=1}^{J} \bar{k}_j\, \bar{N}_j - \frac{\beta\, N}{z}, \;\; K_s = \frac{\beta\, N}{z} - \beta\, \bar{\beta}_i \sum_{j=1}^{J} \bar{k}_j\, \bar{N}_j, \tag{A15}$$

where $\beta \equiv 1/\left(\bar{\beta}_s - \bar{\beta}_i\right)$. Substitute K_s in (A15) into (A11)

$$\sum_{j=1}^{J} \left(\tilde{g}_j + \frac{g_0\, \beta\, \tilde{\varphi}_j}{z} + \beta\, \bar{\beta}_i\, g_0\, \bar{N}_j \right) \bar{k}_j = \frac{g_0\, \beta\, \tilde{\varphi}_0}{z} - g - \hat{\varphi}_0\, \phi, \tag{A16}$$

where

$$\hat{\varphi}_0 \equiv \xi_0 + \hat{\varphi} \frac{g_0 \beta}{z}.$$

From (A16) we have

$$\phi_1 \bar{k}_1 = \phi_0 - \hat{\varphi}_0 \phi - \sum_{j=2}^{J} \phi_j \bar{k}_j, \tag{A17}$$

where

$$\phi_0\left(z, \phi, \{\bar{k}_j\}\right) \equiv \frac{g_0 \beta \tilde{\varphi}_0}{z} - g, \quad \phi_j(z) \equiv \tilde{g}_j + \frac{g_0 \beta \tilde{\varphi}_j}{z} + \beta \bar{\beta}_i g_0 \bar{N}_j .$$

in which $\{\bar{k}_j\} \equiv (\bar{k}_2, \dots \bar{k}_J)$.

From (16) we get

$$\phi N_0 = \sum_{j=1}^{J} \left(\phi_{wj} T_j w_j + \phi_{cj} p c_j + \phi_{kj} \bar{k}_j\right) \bar{N}_j . \tag{A18}$$

Substitute (11) into (A18)

$$\phi N_0 = \sum_{j=1}^{J} \left(\bar{\sigma}_j \bar{y}_j + \phi_{kj} \bar{k}_j\right) \bar{N}_j , \tag{A19}$$

where $\bar{\sigma}_j \equiv \phi_{wj} \sigma_j + \phi_{cj} \xi_j$. Insert (A4) in (A19)

$$\phi = n_0 \sum_{j=1}^{J} \left(\bar{\sigma}_j g_j + \phi_{kj}\right) \bar{N}_j \bar{k}_j + n_0 \sum_{j=1}^{J} \bar{\sigma}_j \bar{g}_j \bar{N}_j , \tag{A20}$$

where

$$n_0 = \left(N_0 - \sum_{j=1}^{J} \delta_j \bar{\sigma}_j \bar{N}_j \right)^{-1} .$$

Substitute (A20) into (A17)

$$\bar{k}_1 = \Phi\left(z, \{\bar{k}_j\}\right) \equiv \frac{\phi_0 - \hat{\varphi}_0 n_0 \sum_{j=1}^{J} \bar{\sigma}_j \bar{g}_j \bar{N}_j - \sum_{j=2}^{J} \left[\phi_j + \left(\bar{\sigma}_j g_j + \phi_{kj}\right) \hat{\varphi}_0 n_0 \bar{N}_j\right] \bar{k}_j}{\phi_1 + \hat{\varphi}_0 n_0 \left(\bar{\sigma}_1 g_j + \phi_{k1}\right) \bar{N}_1}. \tag{A21}$$

We thus showed the computational procedure in the Lemma. From the procedure, (A21), and (12), we get

$$\dot{\bar{k}}_1\left(z, \phi, \{\bar{k}_j\}, t\right) = \lambda_1 \bar{y}_1 - \Phi, \tag{A22}$$

$$\dot{\bar{k}}_j = \Lambda_j\left(z, \phi, \{\bar{k}_j\}, t\right) \equiv \lambda_j \bar{y}_j - \bar{k}_j, \quad j = 2, ..., J. \tag{A23}$$

Take derivatives of (A21) in t

$$\dot{\bar{k}}_1 = \frac{\partial \Phi}{\partial z}\dot{z} + \frac{\partial \Phi}{\partial t} + \sum_{j=2}^{J} \Lambda_j \frac{\partial \Phi}{\partial \bar{k}_j}, \tag{A24}$$

where we apply (A23). Equations (A24) and (A22) imply

$$\dot{z} = \left(\lambda_1 \bar{y}_1 - \Phi - \frac{\partial \Phi}{\partial t} - \sum_{j=2}^{J} \Lambda_j \frac{\partial \Phi}{\partial \bar{k}_j}\right)\left(\frac{\partial \Phi}{\partial z}\right)^{-1}. \tag{A25}$$

By (A23) and (A25) we have the differential equations in the Lemma.

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